JOURNAL OF APPROXIMATION THEORY **30**, 53–58 (1980)

The Order of Magnitude of Unbounded Functions and Their Degree of Approximation by Piecewise Interpolating Polynomials

J. S. BYRNES

Department of Mathematics, University of Massachusetts, Boston, Massachusetts 02125

AND

O. SHISHA

Department of Mathematics, University of Rhode Island, Kingston, Rhode Island 02881

Received October 12, 1978

1. Our purpose is to relate the order of magnitude of real functions $f(x)$ as $x \to 0^+$ to their degree of approximation by piecewise polynomials interpolating them on some simple denumerable sets of points. A similar relation, for functions on $[1, \infty)$, is given in [1].

2. Let f be a real function on $(0, 1]$ and let k be a positive integer. For every $a \in [\frac{1}{2}, 1]$ we denote by $P_{a,k}(f, x) \equiv P_{a,k}(f)$ the function with domain $(0, a]$ which in each

$$I_{a,n} = (a/n, a/(n-1)], \qquad n = 2, 3, \ldots, \tag{1}$$

coincides with the polynomial of degree $\leqslant k$ interpolating f at the $k+1$ equally spaced points

$$x_j = (a/n) + (d_n/k)\, j, \qquad j = 0, 1, \ldots, k, \tag{2}$$

where d_n is the length of $I_{a,n}$. In particular, $P_{a,1}(f)$ is a polygonal function, interpolating f at a/n, $n = 1, 2, \ldots$. In the following theorem we relate the order of magnitude of $f(x)$ as $x \to 0^+$ to that of our "degree of approximation"

$$\langle f \rangle_{k,\delta} = \sup_{1/2 \leqslant a \leqslant 1} \sup_{0 < x \leqslant \delta} |f(x) - P_{a,k}(f, x)|$$

as $\delta \to 0^+$.

Had we defined the "degree of approximation" as $\sup_{0<x\leqslant\delta} |f(x) - P_{a,k}(f, x)|$ for some fixed a, say $a = 1$, we would have given undue weight to the behavior of f at the points $1, \frac{1}{2}, \frac{1}{3}, \ldots$. For example, if f is linear on each

0021-9045/80/090053–06$02.00/0

$[1/n, 1/(n-1)]$, $n = 2, 3, \ldots$, the last sup is $= 0$ while f can be of an arbitrarily large order of magnitude as $x \to 0^+$. It is to avoid such a state of affairs that we define $\langle f \rangle_{k,\delta}$ as we do.

In Section 4 we show that, in our theorem (in one direction), $P_{a,k}$ can be replaced by *any* piecewise polynomial of degree $\leqslant k$ whose knots are $\ldots, a/3, a/2, a$, not necessarily one arising from interpolation.

3. THEOREM. *Let* $0 \leqslant \alpha < k + 1$ *and let* $f^{(k+1)}$ *exist and be nondecreasing or nonincreasing in* $(0, 1]$. *Then* $f(x) = O(x^{\alpha-k-1})$ *as* $x \to 0^+$ *iff* $\langle f \rangle_{k,\delta} = O(\delta^\alpha)$ *as* $\delta \to 0^+$.

In Section 5 we show that the monotonicity requirement in the Theorem cannot be removed. In Section 6 we give an example of an f, with $\langle f \rangle_{1,\delta} = O(\delta)$ as $\delta \to 0^+$, which is not even measurable, showing that having such a degree of approximation does not imply any smoothness of the function.

Proof of the theorem. Assume that $f^{(k+1)}$ is nonincreasing in $(0, 1]$ (otherwise, consider $-f$). Let

$$g(x) = f(x) - \sum_{j=0}^{k+1} \frac{f^{(j)}(1)}{j!} (x-1)^j$$

so that $g(1) = g'(1) = \cdots = g^{(k+1)}(1) = 0$ and $g^{(k+1)}(x) = f^{(k+1)}(x) - f^{(k+1)}(1)$. Also $g(x) - f(x) = \psi(x) - [f^{(k+1)}(1)/(k+1)!](x-1)^{k+1}$ where $\psi(x)$ is a polynomial of degree $\leqslant k$ so that

$$P_{a,k}(g-f) = \psi - [f^{(k+1)}(1)/(k+1)!]\, P_{a,k}((x-1)^{k+1}).$$

Let $0 < t \leqslant \delta \leqslant \frac{1}{2}$ and let $a \in [\frac{1}{2}, 1]$. Then t belongs to some $I_{a,n} (n \geqslant 2)$ so that

$$a/[2(n-1)] \leqslant a/n < t \leqslant \delta.$$

Clearly $P_{a,k}((x-1)^{k+1}, t) = (t-1)^{k+1} - \prod_{j=0}^{k} (t - x_j)$, where the x_j are given by (2), and $|\prod_{j=0}^{k} (t - x_j)| < d_n^{k+1} < (4\delta^2)^{k+1}$. Hence $h(t) = g(t) - f(t) - P_{a,k}(g-f, t)$ satisfies

$$|h(t)| \leqslant |f^{(k+1)}(1)/(k+1)!|\,(4\delta^2)^{k+1}.$$

Observe that $g(t) - P_{a,k}(g, t) = f(t) - P_{a,k}(f, t) + h(t)$, which clearly implies that $\langle f \rangle_{k,\delta} = O(\delta^\alpha)$ as $\delta \to 0^+$ iff $\langle g \rangle_{k,\delta} = O(\delta^\alpha)$ as $\delta \to 0^+$. Also $f(x) = O(x^{\alpha-k-1})$ as $x \to 0^+$ iff $g(x) = O(x^{\alpha-k-1})$ as $x \to 0^+$. Therefore we may assume without loss of generality that

$$f^{(j)}(1) = 0, j = 0, 1, \ldots, k+1, \text{ and hence } (-1)^j f^{(k+1-j)} \text{ is } \geqslant 0 \text{ and nonincreasing in } (0, 1] \text{ for } j = 0, 1, \ldots, k+1. \tag{3}$$

Suppose now that M is a number such that

$$\langle f \rangle_{k,\delta} \leqslant M\delta^{\alpha} \text{ for all positive } \delta \leqslant \text{some } \delta_0 \in (0, \tfrac{1}{2}]. \tag{4}$$

Let $0 < x \leqslant \delta_0$. Define the integer n (> 2) and the numbers a and $\tilde{x}$ by (see (2))

$$1/n < x \leqslant 1/(n-1) < 2/n, \qquad a = (n-1)\,x, \quad \tilde{x} = (a/n) + d_n(2k)^{-1}. \tag{5}$$

Then $\frac{1}{2} < a \leqslant 1, 0 < \tilde{x} < x$.

By the remainder theorem for Lagrange interpolation [2, p. 56], using again the notation (2), for some $\xi \in (a/n, a/(n-1))$,

$$\begin{aligned} |f(\tilde{x}) - P_{a,k}(f, \tilde{x})| &= \frac{f^{(k+1)}(\xi)}{(k+1)!} \prod_{j=0}^{k} |\tilde{x} - x_j| \\ &= \frac{f^{(k+1)}(\xi)}{(k+1)!} \left[\frac{a}{2kn(n-1)}\right]^{k+1} \cdot 1 \cdot 3 \cdots (2k-1) \end{aligned} \tag{6}$$

so that

$$0 \leqslant f^{(k-1)}(x) \leqslant f^{(k+1)}(\xi) \leqslant M_k x^{\alpha-2k-2}, \tag{7}$$

where $M_k = M(k+1)!\,(8k)^{k+1}[1 \cdot 3 \cdots (2k-1)]^{-1}$; the first two inequalities are from (3) and the third from (6), (4) and (5).

By (3) and (7), for every $x \in (0, 1]$,

$$0 \leqslant f^{(k+1)}(x) \leqslant f^{(k+1)}(\delta_0 x) \leqslant \mu_k x^{\alpha-2k-2}, \qquad \text{where} \quad \mu_k = M_k \delta_0^{\alpha-2k-2}.$$

Successive integrations, using (3), yield

$$0 \leqslant (-1)^{k+1} f(x) \leqslant \mu_k \left[\prod_{j=1}^{k+1} (k+j-\alpha)^{-1}\right] x^{\alpha-k-1} \qquad \text{throughout } (0, 1]$$

as required.

For the converse, suppose that, for some constant J,

$$|f(x)| \leqslant J x^{\alpha-k-1} \text{ throughout } (0, 1]. \tag{8}$$

Then, for $j = 0, 1, \ldots, k+1$,

$$0 \leqslant (-1)^{k+1-j} f^{(j)}(x) \leqslant C_j x^{\alpha-k-1-j} \text{ throughout } (0, 1], \qquad \text{where } C_j = 2^{j(k-\alpha)-1+[(j+1)(j+2)/2]} J. \tag{9}$$

This is true for $j = 0$ by (3) and (8) and assuming its truth for some j,

$0 \leqslant j \leqslant k$, we have, for every $x \in (0, 1]$ and a proper $y \in (x/2, x)$,

$$-C_j(x/2)^{\alpha-k-1-j} \leqslant (-1)^{k-j} f^{(j)}(x/2) \leqslant (-1)^{k-j+1}[f^{(j)}(x) - f^{(j)}(x/2)]$$
$$= (-1)^{k-j+1}(x/2) f^{(j+1)}(y) \leqslant (-1)^{k-j+1}(x/2) f^{(j+1)}(x)$$

so that $0 \leqslant (-1)^{k+1-(j+1)} f^{(j+1)}(x) \leqslant 2^{-\alpha+k+2+j} C_j x^{\alpha-k-1-(j+1)} = C_{j+1} x^{\alpha-k-1-(j+1)}$.

Let $\frac{1}{2} \leqslant a \leqslant 1$, $0 < x \leqslant \delta \leqslant \frac{1}{2}$. For a proper $n \geqslant 2$, $x \in I_{a,n}$ (see (1)). Using again (2) and the above remainder theorem, we have, for some $\eta \in (a/n, a/(n-1))$,

$$|f(x) - P_{a,k}(f, x)| = [f^{(k+1)}(\eta)/(k+1)!] \prod_{j=0}^{k} |x - x_j|.$$

By (9) with $j = k + 1$,

$$|f(x) - P_{a,k}(f, x)| \leqslant C_{k+1}\eta^{\alpha-2k-2}[a/\{n(n-1)\}]^{k+1}/(k+1)!$$
$$\leqslant C_{k+1}\eta^{\alpha-k-1}(n-1)^{-k-1}/(k+1)!$$
$$\leqslant C_{k+1}(a/n)^{\alpha-k-1}(2/n)^{k+1}/(k+1)!$$
$$\leqslant C_{k+1}(a/n)^{\alpha}\, 4^{k+1}/(k+1)! \leqslant [4^{k+1}C_{k+1}/(k+1)!]\, \delta^{\alpha}.$$

This completes the proof.

4. COROLLARY. *Assume the hypotheses of the Theorem. A necessary and sufficient condition for $f(x)$ to be $O(x^{\alpha-k-1})$ as $x \to 0^+$ is the existence, for each $a \in [\frac{1}{2}, 1]$, of a function $Q_a(x)$ with domain $(0, a]$, continuous there, which in each $I_{a,n}$ of (1) coincides with some polynomial of degree $\leqslant k$ such that*

$$\sup_{1/2 \leqslant a \leqslant 1} \sup_{0 < x \leqslant \delta} |f(x) - Q_a(x)| = O(\delta^{\alpha})$$

as $\delta \to 0^+$.

Proof. Only sufficiency needs proof. Let μ and $\delta_1 (0 < \delta_1 \leqslant \frac{1}{2})$ be numbers such that

$$\sup_{1/2 \leqslant a \leqslant 1} \sup_{0 < x \leqslant \delta} |f(x) - Q_a(x)| \leqslant \mu\delta^{\alpha} \qquad \text{for all } \delta \in (0, \delta_1].$$

Let $0 < t \leqslant \delta \leqslant \delta_1/2$, $a \in [\frac{1}{2}, 1]$ and set

$$R_a(x) = P_{a,k}(f, x) - Q_a(x).$$

For some $n \geqslant 3$, $t \in I_{a,n}$ and, using (2),

$$R_a(t) = \sum_{j=0}^{k} R_a(x_j) \prod_{\substack{s=0 \\ s \neq j}}^{k} (t - x_s)/(x_j - x_s).$$

For $j = 0, 1, \ldots, k$, $0 < x_j \leqslant a/(n-1) < 2a/n < 2t \leqslant 2\delta \leqslant \delta_1$ and hence

$$|R_a(x_j)| \leqslant \mu(2\delta)^\alpha.$$

Therefore $|R_a(t)| \leqslant (k+1)\, k^k \mu(2\delta)^\alpha$ and hence $|f(t) - P_{a,k}(f, t)| \leqslant \mu[1 + (k+1)\, k^k 2^\alpha]\, \delta^\alpha$. Thus $\langle f \rangle_{k,\delta} \leqslant \mu[1 + (k+1)\, k^k 2^\alpha]\, \delta^\alpha$ if $0 < \delta \leqslant \delta_1/2$ and, by our Theorem, $f(x) = O(x^{\alpha-k-1})$ as $x \to 0^+$.

5. We show that the monotonicity requirement in the Theorem cannot be removed. Consider the function $G(x) \equiv x^{-1} + \sin(\pi x^{-1})$, analytic in $(0, 1]$. Take $k = \alpha = 1$. Then $G(x) = O(x^{\alpha-k-1})$ as $x \to 0^+$, but $\langle G \rangle_{k,\delta}$ is not $O(\delta^\alpha)$ as $\delta \to 0^+$. For suppose it is. For every $a \in [\frac{1}{2}, 1]$, $t \in (0, a]$ we have

$$\sin(\pi t^{-1}) - P_{a,1}(\sin(\pi x^{-1}), t) = G(t) - P_{a,1}(G, t) - t^{-1} + P_{a,1}(x^{-1}, t).$$

This readily implies, for every $\delta \in (0, \frac{1}{2}]$,

$$\langle \sin(\pi x^{-1}) \rangle_{1,\delta} \leqslant \langle G \rangle_{1,\delta} + \langle x^{-1} \rangle_{1,\delta}$$

and hence, by our Theorem applied to $f(x) = x^{-1}$,

$$\langle \sin(\pi x^{-1}) \rangle_{1,\delta} = O(\delta) \qquad \text{as} \quad \delta \to 0^+.$$

But since $P_{1,1}(\sin(\pi x^{-1}), t) = 0$, $\langle \sin(\pi x^{-1}) \rangle_{1,\delta} \geqslant 1$ for every $\delta \in (0, \frac{1}{2}]$.

6. We finally construct a real function F on $(0, 1]$ for which $\langle F \rangle_{1,\delta} = O(\delta)$ as $\delta \to 0^+$ but which is measurable in no $(0, \delta)$, $0 < \delta \leqslant 1$.

For $n = 2, 3, \ldots$ let H_n be a nonmeasurable subset of $(1/n, 1/(n-1)]$ and let $H = \bigcup_{n=2}^{\infty} H_n$. For each $x \in (0, 1]$ let $F(x) = x^{-1}$ if $x \notin H$ while, if x lies in some H_n, let $F(x) = x^{-1} + n^{-1}$ so that $|F(x) - x^{-1}| < x$. Taking in our Theorem $f(x) = x^{-1}$, $\alpha = k = 1$, we have $\langle x^{-1} \rangle_{1,\delta} \leqslant 256\delta$ for all $\delta \in (0, \frac{1}{2}]$ (see the end of its proof). Let $0 < t \leqslant \delta \leqslant \frac{1}{2}$, $\frac{1}{2} \leqslant a \leqslant 1$, say $t \in I_{a,n}$. As $a/n < \delta$, $a/(n-1) \leqslant 2a/n$, we have $|F(a/n) - (a/n)^{-1}| < \delta$, $|F(a/(n-1)) - (a/(n-1))^{-1}| < 2\delta$. Hence $|P_{a,1}(F(x) - x^{-1}, t)| < 2\delta$ and therefore

$$\begin{aligned} |F(t) - P_{a,1}(F, t)| &\leqslant |t^{-1} - P_{a,1}(x^{-1}, t)| + |F(t) - t^{-1}| \\ &\quad + |P_{a,1}(F(x) - x^{-1}, t)| < 259\delta. \end{aligned}$$

Hence $\langle F \rangle_{1,\delta} = O(\delta)$ as $\delta \to 0^+$.

Let $0 < \delta \leqslant 1$, and let n be an integer $> 1 + \delta^{-1}$. If F were measurable in $(0, \delta)$, so would be $F(x) - x^{-1}$ in $S = (1/n, 1/(n-1)]$; hence the subset T of S where $F(x) - x^{-1} \neq 0$ would be measurable; but T is the nonmeasurable set H_n.

REFERENCES

1. J. S. BYRNES AND O. SHISHA, The order of magnitude of functions of a positive variable and their degree of approximation by piecewise interpolating polynomials, *in* "Approximation Theory III," Proceedings of a conference held in January 1980 in Austin, Texas (E. W. Cheney, Ed.), Academic Press, New York, to appear.
2. P. J. DAVIS, "Interpolation and Approximation," Blaisdell, New York, 1963; Dover, New York, 1975.